

PATCH IDEALS AND PETERSON VARIETIES

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ABSTRACT. *Patch ideals* encode neighbourhoods of a variety in GL_n/B . For Peterson varieties we determine generators for these ideals and show they are complete intersections, and thus Cohen-Macaulay and Gorenstein. Consequently, we

- combinatorially describe the singular locus of the Peterson variety;
- give an explicit equivariant K -theory localization formula; and
- extend some results of [B. Kostant '96] and of D. Peterson to intersections of Peterson varieties with Schubert varieties.

We conjecture that the tangent cones are Cohen-Macaulay, and that their h -polynomials are nonnegative and upper-semicontinuous.

Similarly, we use patch ideals to briefly analyze other examples of torus invariant subvarieties of GL_n/B , including Richardson varieties and Springer fibers.

1. INTRODUCTION

1.1. Overview. The **Peterson variety** is defined as

$$\text{Pet}_n = \{F_\bullet \in \text{Flags}(\mathbb{C}^n) : N \cdot F_i \subset F_{i+1}\},$$

where N is a fixed but arbitrary regular nilpotent matrix (up to isomorphism, Pet_n is independent of the choice of conjugate of N), and $\text{Flags}(\mathbb{C}^n)$ is the variety of complete flags of subspaces $F_\bullet : \langle 0 \rangle \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n$.

Since its introduction by D. Peterson in the 1990's, in connection to quantum cohomology of flag varieties, this space has attracted significant attention; see, for instance, [Kos96, Rie03, Tym07, HarTym11] and the references therein.

While Peterson varieties are known to be singular for $n \geq 3$ [Kos96, Theorem 6], problems about their local structure exist:

Problem 1.1. *Give an explicit combinatorial description of the singular locus.*

Problem 1.2. *Determine properties (e.g., Cohen-Macaulay, Gorenstein, normal) that hold for all singular points.*

Problem 1.3. *Give an explicit combinatorial rule for discrete local singularity measures (e.g., equivariant K -theory localizations, Hilbert-Samuel multiplicity).*

Such problems raise two meta-questions about varieties X defined inside GL_n/B .

First, what methods are available to study local features of a given $X \subseteq GL_n/B$? This question arises since a number of significant varieties are defined to sit in the flag variety.

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Perhaps most prominent are the Schubert varieties. However, there are also the Richardson varieties, torus orbit closures $\overline{T}x$, and the family of Hessenberg varieties (which include the Springer and Peterson varieties), among others. One desires uniform and flexible approaches for theoretical and computational analysis.

Indeed, important and influential techniques have been developed that are used to study such X . For example, *Standard Monomial Theory*, a survey of which is found in [LakLitMag98], *GKM-theory* [GorKotMac98], *Frobenius splitting*, a recent book for which is [BriKum05], and use of the *Peterson translate* [CarKut03]. These techniques are especially effective, but not necessarily definitive, for Schubert varieties.

Therefore, we revisit this meta-question by examining the general use of a simple choice of local coordinates and equations encoded in what we call the *patch ideal*. We emphasize that in the literature, one finds many examples of similar ideas, used for instances of X (and usually X is a Schubert variety). We provide a brief review in Section 2, where we also offer a synthesis of a selection of these ideas. This facilitates our common discussion of Peterson varieties, specific earlier work on Schubert varieties [WooYon08, WooYon09, LiYon10a, LiYon10b], and some other X . In our main example of Peterson varieties, we show how to apply these ideas to provide some answers to the above problems.

Second, what are some candidate properties for comparing different $X \subseteq GL_n/B$? All of the aforementioned examples of X share the common feature of an action of a (sub)torus of the invertible diagonal matrices $T \subset GL_n$. Deeper commonalities among some of them are known; a partial list of well-studied properties is given in Problem 1.2. We conjecture that a number of interesting X (including Peterson and Schubert varieties) share common properties concerning their tangent cones.

Turning to our main results, we give coordinates and equations that scheme-theoretically cut out a natural open neighbourhood (the *patch*) of any point of $X = \text{Pet}_n$. We prove these equations define a radical ideal (the patch ideal). To do this, we show the ideal is a complete intersection and hence Cohen-Macaulay. Thus, the zero scheme is reduced if it is generically reduced. To prove the latter, we establish the existence of a smooth point.

Consequently, Pet_n is a local complete intersection, and therefore Cohen-Macaulay and Gorenstein. J. Carrell informed us that at least some of these properties (see Corollary 1.8) were known to D. Peterson (unpublished); we also offer a generalization, see Theorem 7.6. In addition, B. Kostant [Kos96, Theorem 14] partially described the singular locus of Pet_n . We give a combinatorial description of the singular locus. We also obtain an explicit equivariant K -theory localization formula, by approaching the question in terms of combinatorial commutative algebra.

1.2. Peterson varieties and the main results. Identify $\text{Flags}(\mathbb{C}^n) \cong GL_n/B$, where we let $B \subset GL_n$ denote the subgroup of invertible upper triangular matrices. A coset gB corresponds to the flag F_\bullet where F_i is the subspace spanned by the vectors represented by the leftmost i columns of any coset representative of gB .

Let $w = w_P$ be the maximal Coxeter length element for the parabolic (Young) subgroup

$$(1.1) \quad W_P \cong S_{i_1} \times S_{i_2} \times \cdots \times S_{i_k}, \quad i_1 + i_2 + \cdots + i_k = n$$

of the symmetric group S_n , in the latter's role as the Weyl group W of GL_n . Let

$$(1.2) \quad U_P = B^{(1)} \oplus B^{(2)} \oplus \cdots \oplus B^{(k)} \subset U$$

be the $(n-k)$ -dimensional subgroup of unipotent upper triangular matrices U , consisting of block matrices where the block $B^{(j)}$ is an $i_j \times i_j$ upper triangular unipotent matrix with equal entries on each superdiagonal (see Example 4.1). There is a decomposition

$$(1.3) \quad \text{Pet}_n = \coprod_{w_P} \mathcal{J}_{w_P, \text{Pet}_n},$$

of Pet_n into affine cells indexed by the parabolic subgroups P contained in B , where

$$(1.4) \quad \mathcal{J}_{w_P, \text{Pet}_n} = U_P \cdot w_P B \cong \mathbb{C}^{n-k};$$

see Lemma 4.2(IV).

Our first result describes the singular locus of Pet_n . This answers Problem 1.1:

Theorem 1.4. *The singular locus of Pet_n is given by*

$$\text{Sing}(\text{Pet}_n) = \coprod_{w_P} \mathcal{J}_{w_P, \text{Pet}_n},$$

where the union is over all w_P satisfying any of the equivalent conditions (I)–(III) below:

- (I) $w_P B$ is singular in Pet_n ;
- (II) w_P is not one of the permutations (written in one-line notation):
 - $n \ n-1 \ n-2 \ \cdots \ 3 \ 2 \ 1$ (i.e., the maximal Coxeter length element $w_0 \in S_n$);
 - $1 \ n \ n-1 \ \cdots \ 3 \ 2$; or
 - $n-1 \ n-2 \ \cdots \ 1 \ n$.
- (III) w_P contains at least one of the patterns 123 or 2143.

If it were true that every U_P acts on Pet_n , the condition (I) would be obvious (see Lemma 2.2(I)). However, this is not actually the case. Although (II) \Leftrightarrow (III) is easy, the pattern criterion (III) is reminiscent of the results about singularities of Schubert varieties; see [BilLak00, Chapter 8] and the references therein.

The one-dimensional torus $S = \{s(t) = \text{diag}(t, t^2, \dots, t^n) : t \in \mathbb{C}^\star\} \subseteq T$ acts on Pet_n (see Lemma 4.2(I)). Thus, there is an S -equivariant K -theory class $[\mathcal{O}_{\text{Pet}_n}] \in K_S(GL_n/B)$, where $\mathcal{O}_{\text{Pet}_n}$ is the (S -equivariant) structure sheaf of Pet_n . It makes sense to localize this class at the S -fixed points $w_P B$. This localization satisfies

$$[\mathcal{O}_{\text{Pet}_n}]|_{w_P B} \in K_S(w_P B) \cong \mathbb{Z}[\chi^{\pm 1}]$$

where $\chi : S \rightarrow \mathbb{C}^\star$ is the character $s(t) \mapsto t$ of S .

Theorem 1.5. *The following localization formula holds:*

$$[\mathcal{O}_{\text{Pet}_n}]_S|_{w_P B} = \prod_{j=1}^{n-2} \prod_{k \neq w_P(i), 1 \leq i \leq j+1} (1 - \chi^{k+1-w_P(j)}) \in K_S(w_P B) \cong \mathbb{Z}[\chi^{\pm 1}].$$

Theorem 1.5 addresses Problem 1.3 for only one of the available numerical local invariants. For example, we do not know a formula for the Hilbert-Samuel multiplicities of Pet_n . However, we will be able to at least compute some of them (using (P.V) of Section 2). This has been effective enough to support some new conjectures about Peterson varieties.

We will deduce Theorems 1.4 and 1.5 from our next result, providing local coordinates and equations for Pet_n . Let $Z^{(w_P)}$ denote the generic matrix in $w_P U_-$ where U_- is the

subgroup of unipotent matrices in the group of invertible lower triangular matrices B_- . This matrix has z_{ij} placed in the usual matrix coordinate position and

$$(1.5) \quad z_{w_P(j),j} = 1 \text{ and } z_{ij} = 0 \text{ if } j > w_P^{-1}(i).$$

Let $\mathbf{z} = \{z_{ab}\}$ be the collection of variables not specialized by (1.5). Also, let $Z_j = [z_{1j} \ z_{2j} \ \cdots \ z_{nj}]^t$ denote the j^{th} column of $Z^{(w_P)}$. See Example 1.7 below.

Fix N to be the standard regular nilpotent $n \times n$ matrix consisting of 1's on the main superdiagonal and 0's elsewhere. Consider the following system of linear equations in the unknowns $\{\alpha_{j,\ell}\}$:

$$(1.6) \quad N \cdot Z_j = \alpha_{j,1}Z_1 + \alpha_{j,2}Z_2 + \cdots + \alpha_{j,j+1}Z_{j+1}, \quad \text{for } 1 \leq j \leq n-2.$$

For each j , label the equation in the k -th row of (1.6) by

$$(1.7) \quad f_{k,j} : \alpha_{j,1}z_{k,1} + \cdots + \alpha_{j,j+1}z_{k,j+1} = z_{k+1,j},$$

where $1 \leq j \leq n-2$ and $1 \leq k \leq n$. We use the convention $z_{n+1,j} = 0$.

Solving for the unknowns $\alpha_{j,1}, \alpha_{j,2}, \dots, \alpha_{j,j+1} \in \mathbb{C}[\mathbf{z}]$, in that order, it is straightforward to check that the subsystem of (1.6), consisting of equations $f_{k,j}$ where $k = w_P(\ell)$ and $1 \leq \ell \leq j+1$, has a unique solution, recursively expressed as:

$$(1.8) \quad \alpha_{j,\ell} = z_{w_P(\ell)+1,j} - \left(\sum_{t=1}^{\ell-1} \alpha_{j,t} z_{w_P(\ell),t} \right) \in \mathbb{C}[\mathbf{z}].$$

Consider the $\binom{n-1}{2}$ equations $f_{k,j}$ where $k \neq w_P(\ell)$ for $1 \leq \ell \leq j+1$. Let

$$g_{k,j} = \alpha_{j,1}z_{k,1} + \cdots + \alpha_{j,j+1}z_{k,j+1} - z_{k+1,j} \in \mathbb{C}[\mathbf{z}]$$

be the polynomial obtained from $f_{k,j}$ after substituting for $\alpha_{j,\ell}$ with the unique solution given by (1.8). Define the ideal

$$(1.9) \quad I_{w_P, \text{Pet}_n} = \langle g_{k,j} : k \neq w_P(\ell), 1 \leq \ell \leq j+1 \rangle \subseteq \mathbb{C}[\mathbf{z}]$$

and set $Y_{w_P, \text{Pet}_n} = \text{Spec}(\mathbb{C}[\mathbf{z}]/I_{w_P, \text{Pet}_n})$.

The following result, in the terminology of Section 2, shows that I_{w_P, Pet_n} is a patch ideal:

Theorem 1.6. Y_{w_P, Pet_n} is reduced and it is isomorphic to the affine open neighbourhood $\mathcal{N}_{w_P, \text{Pet}_n} = w_P B_- B / B \cap \text{Pet}_n$ of $w_P B \in \text{Pet}_n$.

Example 1.7. Consider the point $w_P B \in \text{Pet}_4$ where

$$w_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and hence } Z^{(w_P)} = \begin{pmatrix} z_{11} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ z_{31} & z_{32} & z_{33} & 1 \\ z_{41} & z_{42} & 1 & 0 \end{pmatrix}.$$

Here $(\alpha_{11}, \alpha_{12}) = (z_{31}, 1 - z_{31}z_{11})$ and $(\alpha_{21}, \alpha_{22}, \alpha_{23}) = (z_{32}, -z_{32}z_{11}, -z_{41}z_{32} + z_{42}z_{32}z_{11})$. Now, g_{31}, g_{41} and g_{32} generate the ideal, i.e.,

$$I_{w_P, \text{Pet}_4} = \langle z_{41} - z_{32} - z_{31}^2 + z_{31}z_{32}z_{11}, z_{41}z_{31} + z_{42} - z_{42}z_{31}z_{11}, \\ z_{42} - z_{31}z_{32} + z_{32}^2z_{11} + z_{41}z_{32}z_{33} - z_{42}z_{32}z_{33}z_{11} \rangle. \quad \square$$

The following provides a partial answer to Problem 1.2:

Corollary 1.8. Pet_n is a local complete intersection (and hence is Cohen-Macaulay and Gorenstein). It is normal if and only if $n \leq 3$.

Proof. The neighbourhoods $\mathcal{N}_{w_P, \text{Pet}_n}$ cover Pet_n (see (P.I) in Section 2). Since $\dim \text{Pet}_n = n - 1$, Theorem 1.6 asserts there is an open neighbourhood of any $gB \in \text{Pet}_n$ isomorphic to an affine variety of codimension $\binom{n-1}{2}$ in $\mathbb{C}^{\binom{n}{2}}$. This codimension equals the number of generators of I_{w_P, Pet_n} . Hence I_{w_P, Pet_n} is a complete intersection, proving the first statement. Since every point is locally Cohen-Macaulay and Gorenstein, the same is true of Pet_n ; see, e.g., [Eis95, Prop 18.13] and [Eis95, Cor 21.19].

The normality statement was proved in [Kos96, Theorem 14]. Now, $\text{Pet}_2 \cong \mathbb{P}^1$ is smooth, whereas one can directly check that Pet_3 is normal at its unique singular point $(id)B$. We recover Kostant's theorem since Theorem 1.4 asserts the singular locus is codimension one when $n \geq 4$: for $w_P = k \ k - 1 \ k - 2 \ \cdots \ 1 \ n \ n - 1 \ \cdots \ k + 1$ and $k \geq 2$, all points in $\mathcal{J}_{w_P, \text{Pet}_n} \cong \mathbb{C}^{n-2}$ are singular. \square

1.3. Organization. In Section 2, we provide background about patch ideals that we will need. In Section 3, we make additional comparisons of our work on Peterson varieties with the literature. In Section 4, we prove Theorem 1.6, after recalling necessary facts about Peterson varieties. In Section 5, we prove Theorem 1.4. In Section 6, we prove Theorem 1.5 and discuss conjectures about the tangent cone. In Section 7, we examine some other $X \subseteq GL_n/B$. We hint that Conjecture 6.7 from Section 6 may be more broad. We will also state and prove Theorem 7.6 that generalizes Theorem 1.6 and Corollary 1.8 to Peterson-Schubert varieties.

2. BACKGROUND, ANALYSIS USING PATCH IDEALS (P.I)–(P.V)

We present five rudimentary themes we use to study $X \subseteq GL_n/B$. These apply standard facts about the flag variety [BilLak00, Bri05], equivariant K -theory [ChrGin97], and combinatorial commutative algebra [Eis95, MilStu04], summarized for our setting.

(P.I) *A choice of open neighbourhood of X :* Recall the two **Bruhat decompositions**

$$(2.1) \quad GL_n/B = \coprod_{w \in S_n} BwB/B = \coprod_{w \in S_n} B_-wB/B.$$

The opposite big cell B_-B/B provides an affine open neighbourhood of $(id)B$ in GL_n/B . Therefore

$$\mathcal{N}_{g,X} = gB_-B/B \cap X$$

provides an affine open neighbourhood of gB in X . This may be called an X **patch** (we thank A. Knutson for the terminology).

We desire explicit coordinates and equations for $\mathcal{N}_{g,X}$. By the first Bruhat decomposition (2.1), $gB = (bw)B$ for some $b \in U$ and $w \in S_n$. In fact, we may further assume $w^{-1}bw \in U_-$ [Hum64, Theorem 28.4]. So we may assume $g = bw$ for such b in what follows. Now, consider the projection map

$$\pi : GL_n \rightarrow GL_n/B.$$

Let U_- be the subgroup of unipotent lower triangular matrices. Since π is a trivial fibration over B_-B/B with fiber B , it admits a local section

$$\sigma : B_-B/B \rightarrow GL_n$$

such that $\sigma(B_-B/B) = U_- \subset GL_n$ [Bri05, Section 1.2]. Similarly, π admits a local section

$$\sigma_g := g\sigma g^{-1} : gB_-B/B \rightarrow GL_n,$$

which provides a scheme-theoretic isomorphism $\sigma_g(gB_-B/B) = gU_- \subset GL_n$, and hence each section σ_g identifies explicit coordinates for the patch $\mathcal{N}_{g, GL_n/B} = gB_-B/B$. Restricting π and σ_g to X one obtains:

$$(2.2) \quad \mathcal{N}_{g,X} = gB_-B/B \cap X \cong \sigma_g(gB_-B/B \cap X) \cong \pi^{-1}(X) \cap \sigma_g(gB_-B/B),$$

scheme-theoretically (the final intersection being transverse).

A generic matrix $Z^{(w)}$ in wU_- has 1's in row $w(j)$ and column j , and in each row 0's to the right of the 1. Let us index the unspecialized entry of $Z^{(w)}$ in row i and column j by z_{ij} . Hence a generic matrix $Z^{(bw)}$ in bwU_- is $bZ^{(w)}$, i.e., a matrix with affine linear forms in the indeterminates $\mathbf{z} = \{z_{ij}\}$. This explicates the fact $bwU_- = wU_-$. We thus identify

$$(2.3) \quad \mathbb{C}[bwU_-] \cong \mathbb{C}[\mathbf{z}]$$

In view of the identifications (2.2) and (2.3), we may interpret $\mathcal{N}_{g,X}$ as subscheme of $\sigma_g(gB_-B/B) \cong bwU_- = wU_- \cong \mathbb{C}^{\binom{n}{2}}$.

It follows that $\mathcal{N}_{bw,X} = \mathcal{N}_{w,X}$. Moreover, since the patches $\{wB_-B/B : w \in S_n\}$ cover GL_n/B , $\{\mathcal{N}_{w,X} : wB \in X\}$ covers $X \subseteq GL_n/B$. Thus, it is mostly superfluous to define $\mathcal{N}_{g,X}$ for $g \neq w$. However, to conduct some local analysis near gB , we set up coordinates/equations where the origin corresponds to $gB = (bw)B$:

Definition 2.1. The **patch ideal** is $I_{g,X} = I(\mathcal{N}_{g,X}) \subseteq \mathbb{C}[\mathbf{z}]$.

Specifically for $I_{w,X}$, the action of $T \subseteq GL_n$ on wB_-B/B induces an action on wU_- , namely scale each row of a matrix independently, and then rescale each column so that the entry in row $w(j)$ and column j remains a 1. This induces a grading on $\mathbb{C}[wU_-] \cong \mathbb{C}[\mathbf{z}]$, which in our coordinates assigns

$$(2.4) \quad \deg(z_{ij}) = t_i - t_{w(j)}.$$

If $S \subseteq T$ is an algebraic subtorus that acts on X , the grading on $I_{w,X}$ coming from S is a coarsening of (2.4).

(P.II) *Using generators for the patch ideal:* We will assume X is reduced. If X is described as a collection of flags, it is “typically easy” (this is made precise in our examples) to find equations that define an ideal that set-theoretically cuts out $\mathcal{N}_{g,X}$. One does this by determining the equations defining $\pi^{-1}(X) \subseteq GL_n$ and demanding that the matrix entries (thought of as coordinates in $\mathbb{C}[\mathbf{z}]$) of $Z^{(bw)} = bZ^{(w)} \in GL_n$ from (P.I) satisfy these equations. It remains to show these equations generate a radical ideal. Two typical ways to guarantee this, and that we refer to, are:

(i): If the generators define a complete intersection, the associated scheme is Cohen-Macaulay. Thus if the scheme is generically reduced, it is reduced (see [Eis95, Exercise 18.9]). Generic reducedness is guaranteed if X is irreducible and has a smooth point.

(ii): If the generators form a Gröbner basis with squarefree lead terms, the initial ideal is radical, and so must have been the original ideal (see [Eis95, Section 15.8]).

In our computational analysis from Sections 6 and 7 we use facts about free resolutions to verify properties of X ; see e.g., [BruHer93, Eis95]. A point gB is Cohen-Macaulay if the minimal free resolution of the local ring $(\mathbb{C}[\mathbf{z}]/I_{g,X})_{(z_{ij})}$ is **short**, i.e., it has length

equal to the codimension of $\mathcal{N}_{g,X}$ inside $\mathbb{A}^{\binom{n}{2}}$. It is also Gorenstein if the resolution is also **symmetric**, i.e., the betti numbers are symmetric, see e.g., [Eis95, Chapter 21] and specifically Corollary 21.16 contained therein. In addition, if $S \subseteq T$ acts on X we can speak of the S -graded finite free resolution:

$$(2.5) \quad 0 \rightarrow F_N \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{C}[\mathbf{z}]/I_{w,X} \rightarrow 0,$$

If this resolution is short (respectively, also symmetric) then $\mathcal{N}_{w,X}$ is Cohen-Macaulay (respectively Gorenstein).

(P.III) *A decomposition of X :* From (2.1) one inherits a decomposition

$$X = \coprod_w \mathcal{J}_{w,X} \text{ where } \mathcal{J}_{w,X} = (BwB/B) \cap X.$$

Recall that a function $f : X \rightarrow \mathbb{R}$ is **upper-semicontinuous on X** if for any $c \in \mathbb{R}$, the set $\{p \in X \mid f(p) \geq c\}$ is closed. In good cases, one can compare points of $\mathcal{J}_{w,X}$.

Lemma 2.2. *Suppose L is a linear algebraic group that acts on X .*

- (I) *If L acts transitively on $\mathcal{J}_{w,X}$, there is a local isomorphism between any two points in $\mathcal{J}_{w,X}$;*
- (II) *If $wB \in \overline{L \cdot gB}$ for all $gB \in \mathcal{J}_{w,X}$ and $f : X \rightarrow \mathbb{R}$ is L -invariant and upper-semicontinuous on X , then this function is maximized on $\mathcal{J}_{w,X}$ at wB .*

Proof. For (I), the elements of L provide the isomorphisms. The claim (II) follows from the assumption of L -invariance and the definition of upper-semicontinuity. \square

In general, the $\mathcal{J}_{w,X}$ need not be smooth (see, e.g., the case of Richardson varieties in Section 7.2). The reader may wish to compare [Bia73].

(P.IV) *Localization of the equivariant K -class of X :* Let S be an r -dimensional (sub)torus of T acting on GL_n/B . For any such (sub)torus we have the standard identification $\Lambda = \text{Hom}(S, \mathbb{C}^*) \cong \mathbb{Z}^r$. Let $\chi_1 = e^{\lambda_1}, \chi_2 = e^{\lambda_2}, \dots, \chi_r = e^{\lambda_r}$ be the characters of S corresponding to a choice of basis $\lambda_1, \dots, \lambda_r$ of \mathbb{Z}^r .

Consider $\mathbf{M} = \mathbb{C}[\mathbf{z}]/I_{w,X}$ as a $\mathcal{O}(wB_-B/B) \cong \mathbb{C}[\mathbf{z}]$ -module. When $g = w$, we may suppose that (2.5) is a finitely S -graded resolution, where each free module is of the form

$$F_i \cong \mathbb{C}[\mathbf{z}](-b_{i1}) \oplus \cdots \oplus \mathbb{C}[\mathbf{z}](-b_{ir}).$$

Then one can match our choice of characters and the grading so that

$$k(\mathbf{M}) = \sum_{i=0}^N (-1)^i \chi_1^{b_{i1}} \chi_2^{b_{i2}} \cdots \chi_r^{b_{ir}},$$

is the K -polynomial of \mathbf{M} . This polynomial always exists, and is independent of the choice of free resolution [MilStu04, Theorem 8.34].

Let $K_S(GL_n/B)$ be the free abelian group generated by isomorphism classes of S -equivariant vector bundles on GL_n/B . (The smoothness of GL_n/B implies that there is a $\text{Rep}(S)$ -module isomorphism that identifies $K_S(GL_n/B)$ with the Grothendieck (K -homology) group $K^S(GL_n/B)$ of S -equivariant coherent sheaves on GL_n/B .)

Suppose $wB \in (GL_n/B)^T \cap X$. Then $K_S(wB)$ is identified with representation ring of S , i.e., the group algebra

$$\text{Rep}(S) = \mathbb{Z}[\Lambda] = \bigoplus_{i=1}^r \mathbb{Z} \cdot \chi_i.$$

$K_S(GL_n/B)$ has the structure of a $\text{Rep}(S)$ -module. The map $\rho : \{wB\} \hookrightarrow GL_n/B$ gives rise by pullback to the restriction map $\rho^* : K_S(GL_n/B) \rightarrow \text{Rep}(S)$.

If $X \subseteq GL_n/B$ is an S -stable subscheme, let $[\mathcal{O}_X]$ denote the class of its (S -equivariant) structure sheaf. If $wB \in X$, we use common convention by writing the **localization class**

$$[\mathcal{O}_X]|_{wB} = \rho^*([\mathcal{O}_X]) \in \text{Rep}(S) \cong K_S(wB).$$

Then

$$(2.6) \quad [\mathcal{O}_X]|_{wB} = k(\mathbf{M});$$

see [MilStu04, Proposition 8.23].

In situations such as (P.II)(i) and (ii), one can compute $k(\mathbf{M})$ without explicitly knowing the resolution.

Since (2.1) is an S -invariant CW-decomposition of GL_n/B , it follows that GL_n/B is an equivariantly formal variety; see [GorKotMac98, Theorem 14.1]. This implies that to determine the class of an S -invariant $X \subseteq GL_n/B$ in $K_S(GL_n/B)$ it suffices to determine its localization at each $wB \in X \cap (GL_n/B)^T$. See [KnuRos99] and references therein.

(P.V) *The (projectivized) tangent cone:* Let $(\mathcal{O}_{p,X}, \mathfrak{m}_p)$ denote the local ring of $p \in X$. Then

$$\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,X} = \bigoplus_{i \geq 0} \mathfrak{m}_p^i / \mathfrak{m}_p^{i+1}$$

is the **associated graded ring**. The **projectivized tangent cone** is $\text{Proj}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,X})$. The **Hilbert-Samuel multiplicity** of $p \in X$ is defined to be $\text{mult}_p(X) = \deg(\text{Proj}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,X}))$. This gives useful data about the singular structure of p . However, more refined data is available from the h -**polynomial** $h_{p,X}(\eta)$, which is the numerator of the Hilbert series:

$$\text{Hilb}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,X}, \eta) = \frac{h_{p,X}(\eta)}{(1 - \eta)^{\dim(X)}}.$$

One can compute these invariants of X for $p = gB$ from the generators for $I_{g,X}$: Homogenize the generators $\{g_k\}$ by t . Now pick any term order \prec that favors terms with largest degrees of t first. Compute a Gröbner basis with respect to \prec and set $t = 1$ in each generator. The lowest degree forms of the resulting polynomials define the ideal $I'_{g,X} \subset \mathbb{C}[\mathbf{z}]$, where $I'_{g,X}$ (scheme-theoretically) cuts out $\text{Spec}(\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p,X})$. See [Eis95, Proposition 15.28].

Thus, $h_{gB,X}(\eta)$ is the numerator of the Hilbert series of $\mathbb{C}[\mathbf{z}]/I'_{g,X}$ (when expressed with denominator $(1 - \eta)^{\dim(X)}$). It is also true that the Hilbert-Samuel multiplicity is given by $h_{gB,X}(1)$; see, e.g., [KreRob05, Theorem 5.4.15]. Since $I'_{g,X}$ has the standard grading, one can check if it is Cohen-Macaulay or Gorenstein by computing a minimal free resolution for $\mathbb{C}[\mathbf{z}]/I'_{g,X}$ and checking if it is short (respectively, also symmetric); cf. (P.II).

Example 2.3. Let $X = \text{Pet}_n$. The ideal I_{w_P, Pet_n} in Theorem 1.6 is the patch ideal from (P.I). The “typically easy” determination of the equations defining I_{w_P, Pet_n} up to radical (P.II) is the content of the computations (1.6), (1.7) and (1.8) that pull back the definition

of the Peterson variety along $\pi : GL_n \rightarrow GL_n/B$. We will use (P.II)(i) to prove they actually define I_{w_P, Pet_n} as a radical ideal, and hence $Y_{w_P, \text{Pet}_n} \cong \mathcal{N}_{w_P, \text{Pet}_n}$ (as schemes). The decomposition (1.3) agrees with (P.III); it just happens here that $\mathcal{J}_{w, \text{Pet}_n}$ is a group orbit.

If $gB \in \text{Pet}_n$ is any point then by (1.3) and (1.4), we may suppose $gB \in \mathcal{J}_{w_P, \text{Pet}_n}$ where $g = bw_P$ for $b \in U_P$. Re-centering so the origin is at gB corresponds to the coordinate change $z_{i,j} \mapsto z_{i,j} + \sum_{i < k} b_{i,k} \cdot z_{k,j}$, where $b = (b_{ij}) \in U_P$. Therefore, we may take $I_{g, \text{Pet}_n} \subseteq \mathbb{C}[z]$ to be generated by $\{g_{k,j}\}$ (after this affine linear substitution into each $g_{k,j}$). Thus, Theorem 1.6 trivially extends to I_{g, Pet_n} .

Theorem 1.5 is proved via (P.IV). We develop Conjecture 6.7 using (P.V). \square

Example 2.4. The use of patches to study Schubert varieties of GL_n/B appears throughout the literature, see, e.g., [KazLus79, Knu08] and the references therein. It being so ubiquitous an idea, we do not know a primary source. However, even for Schubert varieties, explicit study of the patch *ideal* seems relatively recent; it appears at least implicitly in [Ful91] with more work in [KnuMil05].

Closely related is the use of the Kazhdan-Lusztig ideal, which is mostly the same as the patch ideal (P.I); thus the themes of Section 2 apply. An explicit Gröbner basis was given in [WooYon09] (after the Gröbner degeneration of [Knu08], see also [Knu09]); cf. (P.II)(ii). The decomposition of (P.III) is the Bruhat decomposition. Combinatorial formulas for their localizations (i.e., specializations of *Grothendieck polynomials*) are geometrically explained via the point of view (P.IV). Combinatorial analysis of the projectivized tangent cone in [LiYon10a, LiYon10b] illustrates (P.V). Also, see Section 7.2. \square

Remark 2.5. Suppose $Y \subseteq \mathbb{P}^n$ is any projective variety, then let $\rho : GL_n/B \rightarrow \mathbb{P}^n$ be the map that projects to the first linear subspace of a flag. This is a smooth morphism, and in fact a locally trivial fibration. Thus, local features of Y can, *in principle*, be studied by considering $X = \rho^{-1}(Y) \subseteq GL_n/B$. \square

Remark 2.6. If $X \subseteq G/P$, then take $X \cap gB_-P/P$ to be an X patch. See work of A. Fink-D. Speyer [FinSpe10] where the equivariant K -localizations of \overline{Tx} in the Grassmannian are connected to matroid invariants.

Separately, see thesis work of M. Snider [Sni11] where the patch ideal of the *positroid variety* in the Grassmannian is computed and the scheme is related to certain Kazhdan-Lusztig varieties (essentially also patch ideals) of the affine Grassmannian. See earlier, related work of A. Knutson-T. Lam-D. Speyer [KnuLamSpe10].

At least for the classical groups G , it is not difficult to extend the constructions (P.I)–(P.V) to attempt to understand $X \subseteq G/B$. Briefly, one imposes additional quadric equations that account for the relevant bilinear form. \square

3. FURTHER COMPARISONS WITH EARLIER WORK ON PETERSON VARIETIES

D. Peterson introduced and studied the strata $\mathcal{Y}_{w_P} = \text{Pet}_n \cap B_-w_P B/B$, where P^c is the parabolic subgroup for the nodes of the Dynkin diagram complementary to those defining P . (One has $\text{Pet}_n = \coprod_P \mathcal{Y}_{w_P}$.) Although these strata are reducible (for examples of this in the Grassmannian case, see [Rie01]), he announced that they are in fact reduced. He furthermore gave a remarkable connection between \mathcal{Y}_{w_P} and quantum cohomology of the partial flag varieties GL_n/P . That is,

$$(3.1) \quad \mathbb{C}[\mathcal{Y}_{w_P}] \cong QH^*(GL_n/P);$$

see [Pet97]. B. Kostant [Kos96] had proved the case \mathcal{Y}_{w_B} of this isomorphism.

J. Tymoczko [Tym07] gave a paving by affine spaces for regular nilpotent Hessenberg varieties. The cells in the case of Pet_n are $\mathcal{J}_{w_P, \text{Pet}_n} = \text{Pet}_n \cap Bw_PB/B$; see (1.4) and Lemma 4.2(IV).

In general $\mathcal{N}_{w_P, \text{Pet}_n}$, \mathcal{Y}_{w_P} , and $\mathcal{J}_{w_P, \text{Pet}_n}$ differ. In particular, the latter two do not give affine neighbourhoods of the T -fixed points of Pet_n . However, $\mathcal{N}_{id, \text{Pet}_n} = \mathcal{Y}_{w_B}$.

K. Rietsch [Rie03] proved Peterson's theorem (3.1) in type A , using results of S. Fomin-S. Gelfand-A. Postnikov [FomGelPos97] and I. Ciocan-Fontanine [Cio99]. With a variation on the coordinates and equations we use for $\mathcal{N}_{g, \text{Pet}_n}$ one can study \mathcal{Y}_{w_P} for the classical groups G , in connection to $QH^*(G/P)$. Here is a small sample:

Example 3.1. When $n = 3$, the quantum cohomology ring is

$$QH^*(GL_3/B) \cong \frac{\mathbb{C}[x_1, x_2, x_3, q_1, q_2]}{\langle E_1^{(3)}, E_2^{(3)}, E_3^{(3)} \rangle}$$

where $E_1^{(3)} = x_1 + x_2 + x_3$, $E_2^{(3)} = x_1x_2 + x_1x_3 + x_2x_3 + q_1 + q_2$ and $E_3^{(3)} = x_1x_2x_3 + x_1q_2 + x_3q_1$ are the *quantum elementary symmetric polynomials*. This presentation is an instance of a result of A. Givental–B. Kim and of I. Ciocan-Fontanine; see, e.g., [FomGelPos97, Equation (1.2)] and the associated references. Setting $x_3 = -(x_1 + x_2)$, and $q_2 = -(q_1 + x_1x_2 + x_2x_3 + x_1x_3)$, we eliminate two variables to obtain

$$QH^*(GL_3/B) \cong \frac{\mathbb{C}[x_1, x_2, q_1]}{\langle x_1^3 - 2x_1q_1 - x_2q_1 \rangle}.$$

Consider the surjection $\psi : \mathbb{C}[Z^{(id)}] \cong \mathbb{C}[z_{21}, z_{31}, z_{32}] \rightarrow QH^*(GL_3/B)$ defined by

$$z_{21} \mapsto -(x_1 + x_2), \quad z_{31} \mapsto x_1x_2 + q_1, \quad z_{32} \mapsto -x_1.$$

Observe

$$\begin{aligned} x_1^3 - 2x_1q_1 - x_2q_1 &= (-x_1 - x_2)(x_1x_2 + q_1) + [(x_1x_2 + q_1) - (-x_1 - x_2)^2](-x_1) \\ &= z_{21}z_{31} + (z_{31} - z_{21}^2)z_{32}. \end{aligned}$$

By Theorem 1.6, the latter polynomial cuts out $Y_{123, \text{Pet}_3} \cong \mathcal{N}_{123, \text{Pet}_3} = \mathcal{Y}_{w_B}$. It therefore follows that ψ induces $\mathbb{C}[\mathcal{Y}_{w_B}] \cong QH^*(GL_3/B)$. This agrees with (3.1); cf. [Rie03]. \square

4. PROOF OF THEOREM 1.6

4.1. Preliminaries. Let $U' \subset U$ be the group U_P where $W_P = S_n$; see (1.2).

Example 4.1. In GL_6 , U' and U_P for $W_P = S_3 \times S_3$, respectively are:

$$U' = \left\{ \begin{pmatrix} 1 & q_1 & q_2 & q_3 & q_4 & q_5 \\ 0 & 1 & q_1 & q_2 & q_3 & q_4 \\ 0 & 0 & 1 & q_1 & q_2 & q_3 \\ 0 & 0 & 0 & 1 & q_1 & q_2 \\ 0 & 0 & 0 & 0 & 1 & q_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\} \quad U_P = \left\{ \begin{pmatrix} 1 & q_1 & q_2 & 0 & 0 & 0 \\ 0 & 1 & q_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r_1 & r_2 \\ 0 & 0 & 0 & 0 & 1 & r_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

where $q_i, r_i \in \mathbb{C}$ for all i . \square

Recall the one-dimensional torus

$$S = \left\{ s(t) = \begin{pmatrix} t & & & \\ & t^2 & & \\ & & \ddots & \\ & & & t^n \end{pmatrix} : t \in \mathbb{C}^\star \right\} \subset T.$$

We will need the following (known) facts.

Lemma 4.2. (I) S acts on $\text{Pet}_n \subseteq GL_n/B$ by left multiplication.

(II) Similarly U' acts on Pet_n , and $\text{Pet}_n = \overline{U' \cdot w_0 B}$.

(III) Pet_n is irreducible.

(IV) The two definitions of $\mathcal{J}_{w_P, \text{Pet}_n}$ (cf. Section 1.2 and (P.I)) agree:

$$\mathcal{J}_{w_P, \text{Pet}_n} = U_P \cdot w_P B = \text{Pet}_n \cap (B w_P B / B).$$

(V) If $gB \in \mathcal{J}_{w_P, \text{Pet}_n}$, then $\lim_{t \rightarrow \infty} s(t) \cdot gB = w_P B$.

(VI) $wB \in \text{Pet}_n$ if and only if $w = w_P$, where w_P is the maximal Coxeter length element of $W_P = S_{i_1} \times S_{i_2} \times \cdots \times S_{i_k}$. Explicitly,

$$w_P = i_1 \ i_1 - 1 \cdots 3 \ 2 \ 1 \ i_1 + i_2 \ i_1 + i_2 - 1 \cdots i_1 + 1 \cdots \cdots n \ n - 1 \cdots i_1 + \cdots + i_{k-1} + 1.$$

Example 4.3. Let $n = 8$ and $W_P = S_3 \times S_2 \times S_3$. Then the permutation matrix for w_P and the generic matrix $Z^{(w_P)}$ in $w_P U_-$ are given by

$$w_P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Z^{(w_P)} = \begin{pmatrix} z_{11} & z_{12} & 1 & 0 & 0 & 0 & 0 & 0 \\ z_{21} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_{41} & z_{42} & z_{43} & z_{44} & 1 & 0 & 0 & 0 \\ z_{51} & z_{52} & z_{53} & 1 & 0 & 0 & 0 & 0 \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} & z_{67} & 1 \\ z_{71} & z_{72} & z_{73} & z_{74} & z_{75} & z_{76} & 1 & 0 \\ z_{81} & z_{82} & z_{83} & z_{84} & z_{85} & 1 & 0 & 0 \end{pmatrix}. \quad \square$$

Proof of Lemma 4.2: For (I), the claim follows from the definition of Pet_n since $(s(t))^{-1} N s(t) = tN$ for $s(t) \in S$. Similarly, the first sentence of (II) holds since U' centralizes N .

(III) was shown in [Kos96, Theorem 6]. One knows $\dim \text{Pet}_n = n - 1$. Clearly, $\dim(U' \cdot w_0 B) = n - 1$, so the (irreducible) closure of this group orbit is all of Pet_n ; this is (II).

For (IV), flags in $B w_P B$ may be identified with the block matrices $K = K^{(1)} \oplus K^{(2)} \oplus \cdots \oplus K^{(k)} \in GL_n$ where $K^{(j)}$ is the matrix with 1's on the antidiagonal, 0's below, and free entries above. The condition that the given flag is in Pet_n implies that there are equal entries on each super-antidiagonal of each $K^{(i)}$. The set of all such K is equal to $U_P w_P B$.

(V) holds by the second equality of (IV) and the fact that the claim is true of any point of $B w_P B / B$, by elementary considerations.

Lastly for (VI), by definition, $F_\bullet = wB \in \text{Pet}_n$ if and only if $N \cdot F_i \subseteq F_{i+1}$ for $1 \leq i \leq n-2$. The latter happens if and only if $N \cdot \vec{e}_{w(i)} = \vec{e}_{w(i)-1} \in F_{i+1}$ whenever $w(i) \neq 1$ (when $w(i) = 1$, $N \vec{e}_{w(i)} = \vec{0} \in F_{i+1}$ is automatic). Here \vec{e}_j is the j -th standard basis vector. That $w = w_P$ for some P then follows from this, by inducting on the position of the "1" in the permutation matrix of w , from left to right. \square

4.2. Proof of Theorem 1.6: We refer the reader to the discussion of (P.I), (P.II) and Example 2.3 in Section 2: Let $gB \in \mathcal{N}_{w_P, \text{Pet}_n}$. Since $\mathcal{N}_{w_P, \text{Pet}_n} \cong \pi^{-1}(\text{Pet}_n) \cap \sigma_{w_P}(w_P B_- B/B) \cong \pi^{-1}(\text{Pet}_n) \cap w_P U_-$ we may assume g is the unique element of $w_P U_-$ sent to gB under $\pi : GL_n \rightarrow GL_n/B$. Since $gB \in \text{Pet}_n$, it must therefore be true that the j -th column of g is contained in the span of the first $j+1$ columns. Therefore, by construction, the entries of g (considered as coordinates in $\mathbb{C}^{\binom{n}{2}}$) must satisfy the defining equations of I_{w_P, Pet_n} . Hence $Y_{w_P, \text{Pet}_n} \supseteq \mathcal{N}_{w_P, \text{Pet}_n}$. The other containment is the same argument run in reverse. Thus $Y_{w_P, \text{Pet}_n} = \mathcal{N}_{w_P, \text{Pet}_n}$ (set-theoretically), where the latter is viewed as a subvariety of $\mathbb{C}^{\binom{n}{2}}$.

The dimension of Pet_n is $n-1$; the same is true of $\text{Pet}_n \cap w_P \cdot B_- B/B$. Thus the codimension of $\mathcal{N}_{w_P, \text{Pet}_n}$ and Y_{w_P, Pet_n} in $\mathbb{C}^{\binom{n}{2}}$ (as identified with $\sigma_{w_P}(w_P \cdot B_- B/B)$) is $\binom{n-1}{2}$. This is the number of generators of I_{w_P, Pet_n} . Hence Y_{w_P, Pet_n} is a complete intersection (and Cohen-Macaulay) [Eis95, Prop 18.13].

We now show that Y_{w_P, Pet_n} is reduced; cf. (P.II). Since Pet_n is irreducible, $\mathcal{N}_{w_P, \text{Pet}_n} = Y_{w_P, \text{Pet}_n}$ (set-theoretic equality) is irreducible. Thus, by (P.II)(i), it suffices to exhibit a smooth point, which we do with the Jacobian criterion. To achieve this, we will use the following fact, which is immediate by an induction using (1.8):

Lemma 4.4. *Fix $1 \leq s \leq n-2$ and let $1 \leq q \leq s+1$. Then $\alpha_{s,q}$ is a polynomial in $\{z_{t,r} : r \leq s\}$.*

Consider the summand $\alpha_{j,j+1} z_{k,j+1}$ of the generator $g_{k,j}$ of I_{w_P, Pet_n} ; this is the unique summand in $g_{k,j}$ involving $z_{k,j+1}$ by Lemma 4.4. The assumption that $g_{k,j}$ is a generator implies $z_{k,j+1}$ is unspecialized. Lemma 4.4 additionally implies that $z_{k,j+1}$ does not appear in $\alpha_{j,j+1}$. Thus,

$$\frac{\partial g_{k,j}}{\partial z_{k,j+1}} = \frac{\partial(\alpha_{j,j+1} z_{k,j+1})}{\partial z_{k,j+1}} = \alpha_{j,j+1}(\mathbf{z}).$$

Similarly, the variable $z_{k,j+1}$ does not appear in any $g_{k',j}$ for $k' \neq k$, or $g_{k',j'}$ for $j' < j$.

Claim 4.5. *There exists a point*

$$(4.1) \quad p \in Y_{w_P, \text{Pet}_n} \text{ such that } \alpha_{j,j+1}(p) \neq 0 \text{ for all } 1 \leq j \leq n-2.$$

Proof. Consider $H_j = \{F_\bullet \in GL_n/B : N \cdot F_i \subset F_{i+1} \text{ if } i \neq j \text{ and } N \cdot F_j \subseteq F_j\}$. In fact H_j is a regular nilpotent Hessenberg variety, see Section 7.1. It follows from a dimension formula of E. Sommers-J. Tymoczko [SomTym06, Theorem 10.2], which is explicitly stated in [Tym06, Section 9.1], that $\dim H_j = n-2$. Notice that $\mathcal{N}_{w_P, H_j} = Z(\alpha_{j,j+1}) \cap Y_{w_P, \text{Pet}_n}$ (set-theoretically).

Suppose $\Gamma(\mathbf{z}) = \prod_{1 \leq j \leq n-2} \alpha_{j,j+1}(\mathbf{z})$ vanishes identically on Y_{w_P, Pet_n} . Then this means (set-theoretically) that $Y_{w_P, \text{Pet}_n} \subseteq \bigcup_{j=1}^{n-2} \mathcal{N}_{w_P, H_j}$. Hence Y_{w_P, Pet_n} is at most $n-2$ dimensional, a contradiction. Thus, one can choose a point $p \in Y_{w_P, \text{Pet}_n}$ not on the hypersurface defined by $\Gamma(\mathbf{z})$. This point satisfies (4.1). \square

Next, arrange the rows of the Jacobian $\left[\frac{\partial g_{k,j}}{\partial z_{ab}} \right]$ associated to generators $g_{k,j}$ by (say) favoring smaller j first and breaking ties by favoring smaller k . This manifests (via echelon form) that the Jacobian, evaluated at p , has full rank (having established $\binom{n-1}{2}$ linearly independent columns) and thus p is a smooth point of Y_{w_P, Pet_n} . \square

5. PROOF OF THEOREM 1.4

The proof is in two parts. First, we need that any $gB = bw_P B \in \mathcal{J}_{w_P, \text{Pet}_n}$ is singular when w_P is not on the asserted “smooth list” from (II). Next, we prove that for the three w_P on that list, the points $w_P B$ are smooth. Since the point $w_P B$ is the most singular in $\mathcal{J}_{w_P, \text{Pet}_n}$ (by Lemma 2.2(II) and Lemma 4.2(V)), all points in that cell are smooth. This establishes the condition (I) and the equivalence of (I) with (II).

For the first part, recall that if $v \in S_n$ and $w \in S_N$ for $n \leq N$ then w **contains the pattern** v if there exists a choice of **embedding indices** $1 \leq \phi_1 < \phi_2 < \dots < \phi_n \leq N$ such that $w(\phi_1), w(\phi_2), \dots, w(\phi_n)$ are in the same relative order as $v(1), \dots, v(n)$.

It is straightforward to verify the equivalence (II) \Leftrightarrow (III) from this definition. We say that w **avoids** v if no such indices exist.

The following two lemmas are clear from the Lemma 4.2(VI):

Lemma 5.1. *Suppose $w_P B \in \text{Pet}_n$. If w_P contains the pattern 123 then one can choose the embedding indices $\phi_1 < \phi_2 < \phi_3$ so that $w_P(\phi_1) = 1$ and $\phi_2 = \phi_1 + 1$.*

Lemma 5.2. *Suppose that $w_P B \in \text{Pet}_n$ and avoids 123. If w_P contains the pattern 2143 then one can choose the embedding indices $\phi_1 < \phi_2 < \phi_3 < \phi_4$ to satisfy $\phi_3 = \phi_2 + 1$.*

Recall the notation and definitions preceding the definition of I_{w_P, Pet_n} in Section 1. Although we need to prove that all points $bw_P B$ are singular, strictly speaking we only show $w_P B$ is singular, for simplicity of notation. However, in view of the discussion in Example 2.3, the general case is a trivial extension. Specifically, Lemma 5.3, the three claims that follow it, and their proofs, are the exactly same in the general case, where we have made the substitution $z_{ij} \mapsto z_{ij} + \sum_{i < k} b_{ik} z_{k,j}$ from Example 2.3. Thus, we leave the details to the interested reader.

Lemma 5.3. *Suppose ϕ satisfies $w_P(\phi) = 1$. Then none of the $\alpha_{\phi,k}(\mathbf{z})$ for $1 \leq k \leq \phi + 1$ contains a nonzero constant term.*

Proof. Since $w_P(\phi) = 1$, $N \cdot Z_\phi$ is a column vector that only involves zeros and the unspecialized variables $z_{t,\phi}$. Hence $\alpha_{\phi,1}$ satisfies the stated conclusion. The claim follows for all the $\alpha_{\phi,k}$ by (1.8) and induction on k . \square

Suppose w_P contains 123. Let $\phi_1 < \phi_2 < \phi_3$ be the embedding indices from Lemma 5.1: $1 = w_P(\phi_1) < w_P(\phi_2) < w_P(\phi_3)$ with $\phi_2 = \phi_1 + 1$.

Claim 5.4. g_{n,ϕ_1} is one of the defining generators of I_{w_P, Pet_n} .

Proof. From the definitions the assertion is clearly true provided

$$(5.1) \quad n \neq w_P(\ell) \text{ for } 1 \leq \ell \leq \phi_1 + 1 = \phi_2.$$

Suppose (5.1) does not hold. Since $w_P(\phi_1) = 1$, this bad ℓ is not equal to ϕ_1 . Also, $\ell \neq \phi_1 + 1 (= \phi_2)$ since this violates the assumptions about ϕ_1, ϕ_2, ϕ_3 . Hence $1 \leq \ell < \phi_1$. This combined with $w_P(\phi_1) = 1$ and the form of w_P given by Lemma 4.2(VI) implies $w_P = w_0$. However, w_0 does not contain 123, a contradiction. Thus the claim holds. \square

Since by Lemma 5.3 we know $\alpha_{\phi_1,k}(\mathbf{z})$ has no nonzero constant terms and since

$$g_{n,\phi_1}(\mathbf{z}) = \alpha_{\phi_1,1}(\mathbf{z})z_{n,1} + \dots + \alpha_{\phi_1,\phi_1+1}(\mathbf{z})z_{n,\phi_1+1}$$

the minimal degree of any term of $g_{n,\phi_1}(\mathbf{z})$ is two. Thus the row of the Jacobian $\left[\frac{\partial g_{k,j}(\mathbf{z})}{\partial z_{ab}}\right]$ (evaluated at the origin) associated to $g_{n,\phi_1}(\mathbf{z})$ consists only of zeros. As the number of rows in Jacobian equals the codim Y_{w_P, Pet_n} , the rank at that point is strictly smaller than the codimension of $Y_{w_P, \text{Pet}_n} = \mathcal{N}_{w_P, \text{Pet}_n}$ in $\mathbb{C}^{\binom{n}{2}}$. Hence by the Jacobian criterion, the origin is not a smooth point. As the origin corresponds to $w_P B$ in our coordinates, $w_P B$ is singular in Pet_n .

Now suppose w_P contains 2143. In view of our analysis above, we may assume w_P avoids 123. Let $\phi_1 < \phi_2 < \phi_3 < \phi_4$ be the embedding indices from Lemma 5.2. Since w_P avoids 123, its form, as stated in Lemma 4.2(VI) satisfies $k \leq 2$. The containment of 2143 means $k = 2$ and $i_1, i_2 \geq 2$. Hence we can actually assume $w_P(\phi_1) = 2$ and $w_P(\phi_2) = 1$.

Using these assumptions, and an argument such as for (5.4) it is straightforward that

Claim 5.5. $g_{n,\phi_1}(\mathbf{z})$ and $g_{n-1,\phi_2}(\mathbf{z})$ are defining generators of I_{w_P, Pet_n} .

Claim 5.6. The Jacobian $\left[\frac{\partial g_{k,j}(\mathbf{z})}{\partial z_{ab}}\right]$, when evaluated at the origin $\mathbf{0}$, has the property that the row corresponding to $g_{n,\phi_1}(\mathbf{z})$ is a scalar multiple of the row corresponding to $g_{n-1,\phi_2}(\mathbf{z})$.

Proof. Observe that $\alpha_{\phi_1, \phi_1+1}(\mathbf{z}) (= \alpha_{\phi_1, \phi_2}(\mathbf{z}))$ contains the constant term 1 (coming from the 1 at the top of NZ_{ϕ_1}). In addition, all entries in NZ_{ϕ_1} , other than the top one, do not contain a constant term. By induction on $k \geq 1$, $\alpha_{\phi_1, k}(\mathbf{z})$ does not contain a constant term for $1 \leq k \leq \phi_1$. Thus $g_{n,\phi_1}(\mathbf{z})$ contains one linear term (and terms of higher degree), which is z_{n,ϕ_2} (from the lefthand side of (1.7)).

Since $w_P(\phi_2) = 1$, Lemma 5.3 states that all of the $\alpha_{\phi_2, k}(\mathbf{z})$ do not have constant terms. Hence $g_{n-1,\phi_2}(\mathbf{z})$ contains a single linear term (and terms of higher degree), i.e., the z_{n,ϕ_2} term coming from the right hand side of (1.7). Thus, in the column of the Jacobian (evaluated at the origin) associated to z_{n,ϕ_2} in rows $g_{n,\phi_1}(\mathbf{z})$ and $g_{n-1,\phi_2}(\mathbf{z})$ the entry is ± 1 and ∓ 1 respectively. In other columns, the entries in those two rows are zero. \square

We have thus completed the argument for the first part of the proof.

We now turn to the second half of the proof, i.e., checking smoothness at the three points indicated by the list in (II). As mentioned at the beginning of this section, smoothness at these points implies smoothness at all other points of their respective strata.

Case $w_P = w_0$: If $w_0 B$ is singular, then so is every point of $U'w_0 B$ by Lemma 4.2(II) and Lemma 2.2(I) combined. However, as Pet_n is the closure of this orbit, that would imply every point of Pet_n is singular, which is impossible.

Case $w_P = n-1 \ n-2 \ \cdots \ 2 \ 1 \ n$: Fix a total order $<$ on the generators $\{g_{k,j}\}$ by

$$(5.2) \quad (k', j') < (k, j) \text{ if } j' < j \text{ and } (k', j) < (k, j) \text{ if } k' < k.$$

For this w_P ,

$$(5.3) \quad \alpha_{j,j+1} = 1 - \left(\sum_{\ell=1}^j \alpha_{j,\ell} z_{w(j+1),\ell} \right)$$

for all $1 \leq j \leq n-2$; cf. (1.8). Hence, the term $z_{k,j+1}$ appears as a linear term in $g_{k,j}$. It does not appear in any $g_{k',j'}$ where $(k', j') < (k, j)$: if $j' < j$ then this variable appears nowhere

in (1.6); if $j = j'$ and $k' \neq k$ no $\alpha_{j,\ell}$ involves this variable, nor does NZ_j , so neither can $g_{k',j}$. Therefore,

$$(5.4) \quad \left. \frac{\partial g_{k,j}}{\partial z_{k,j+1}} \right|_{\mathbf{z}=0} = 1 \quad \text{and} \quad \left. \frac{\partial g_{k',j'}}{\partial z_{k,j+1}} \right|_{\mathbf{z}=0} = 0 \quad \text{for } (k', j') < (k, j).$$

We thus conclude the rows of the Jacobian (evaluated at the origin) are linearly independent. Hence the point $w_P B$ is smooth.

Case $w_P = 1, n, n-1, \dots, 2$: Isolating the case $j = 1$ first, Lemma 5.3 implies that $\alpha_{1,1}$ and $\alpha_{1,2}$ do not contain any constant terms. Thus, the only linear term in each $g_{k,1}$ is the $z_{k+1,1}$ that comes from NZ_1 in (1.6). Therefore,

$$(5.5) \quad \left. \frac{\partial g_{k,1}}{\partial z_{k+1,1}} \right|_{\mathbf{z}=0} = 1 \quad \text{and} \quad \left. \frac{\partial g_{k',1}}{\partial z_{k+1,1}} \right|_{\mathbf{z}=0} = 0 \quad \text{for } k' \neq k.$$

For $2 \leq j \leq n-2$, (5.3) holds. This means each $g_{k,j}$ with $2 \leq j \leq n-2$ contains $z_{k,j+1}$ as a linear term. As with the case $w_P = n-1, n-2, \dots, 2, 1, n$, $z_{k,j+1}$ does not appear in any $g_{k',j'}(\mathbf{z})$ where $(k', j') < (k, j)$ in the order (5.2). Combining this with (5.5) we see (5.4) holds for $1 \leq j \leq n-2$. Again, the columns of the Jacobian (evaluated at the origin) are linearly independent, and thus $w_P B$ is smooth. \square

6. K -POLYNOMIALS, EQUIVARIANT LOCALIZATION AND h -POLYNOMIALS

6.1. Proof of Theorem 1.5. We will in fact prove:

(6.1)

$$[\mathcal{O}_{\text{Pet}_n}]_S|_{w_P B} = k_{w_P, \text{Pet}_n}(\chi) = \prod_{j=1}^{n-2} \prod_{\ell \neq w_P(i), 1 \leq i \leq j+1} (1 - \chi^{\ell+1-w_P(j)}) \in K_S(w_P B) \cong \mathbb{Z}[\chi^{\pm 1}],$$

where $k_{w_P, \text{Pet}_n}(\chi)$ is the K -polynomial of $\mathbb{C}[\mathbf{z}]/I_{w_P, \text{Pet}_n}$ with respect the \mathbb{Z} -graded coarsening of (2.4), associated to $S \subset T$. This coarsening assigns to the variable z_{ij} the degree

$$(6.2) \quad \deg(z_{ij}) = i - w_P(j).$$

The first equality of (6.1) holds by the discussion of (P.IV) that concludes at (2.6).

We use a standard fact: if $I = \langle g_1, \dots, g_N \rangle \subset \mathbb{C}[x_1, \dots, x_M]$ is a \mathbb{Z} -graded ideal that is a complete intersection, then its K -polynomial equals

$$k(\mathbb{C}[x_1, \dots, x_M]/I; \chi) = \prod_{i=1}^N (1 - \chi^{\deg(g_i)}).$$

This is an easy consequence of, e.g., [MilStu04, Exercise 8.12].

By Theorem 1.6, we can apply the above fact to the generators $\{g_{\ell,j}\}$. It remains to determine the degree of the generator $g_{\ell,j}$ for $1 \leq j \leq n-2$ and $\ell \neq w_P(t)$ for $1 \leq t \leq j+1$. Since $g_{\ell,j}$ is homogeneous with respect to the grading (6.2), we need only compute the degree of any term of $g_{\ell,j}$ and show it is equal to $\ell + 1 - w_P(j)$.

Since $g_{\ell,j}$ is a generator, the definitions imply the summand $\alpha_{j,j+1} z_{\ell,j+1}$ appears. If $w_P(j+1) < n$ then the desired degree is easily established since the unspecialized variable $z_{w_P(j+1)+1,j}$ appears in $\alpha_{j,j+1}$ by (1.8). Only slightly harder is the case $w_P(j+1) = n$ where this variable does not exist. Here, by (1.8), $\alpha_{j,j+1}$ contains the summand $\alpha_{j,1} z_{n,1}$. However

$\alpha_{j,1} = z_{w_P(1)+1,j}$. Therefore we conclude $\alpha_{j,j+1}z_{\ell,j+1}$ contains the term $z_{w_P(1)+1,j}z_{n,1}z_{\ell,j+1}$ which has degree $(w_P(1) + 1 - w_P(j)) + (w_P(n) - w_P(1)) + (\ell - w_P(j + 1)) = \ell + 1 - w_P(j)$. The second equality of (6.1) is thus deduced. \square

6.2. The (projectivized) tangent cone. Theorem 1.4 asserts that for Pet_n , the property of being singular is constant on each $\mathcal{J}_{w_P, \text{Pet}_n}$. Lemma 2.2(II) and Lemma 4.2(V) combined imply Hilbert-Samuel multiplicity is maximized on $\mathcal{J}_{w_P, \text{Pet}_n}$ at $w_P B$.

Question 6.1. Is $\text{mult}_{g, \text{Pet}_n}$, or $h_{g, \text{Pet}_n}(\eta)$, constant on $\mathcal{J}_{w_P, \text{Pet}_n}$?

Let us restrict attention to the h -polynomials $h_{w_P, \text{Pet}_n}(\eta)$. These are tabulated for $n = 4$ in Table 1 below, along with the K -polynomials (which were independently computed using SAGE, and agree with Theorem 1.5). We state the following easy facts:

Lemma 6.2. (I) *There is a local isomorphism between any two points of the orbit $(U' \rtimes S) \cdot gB$.*
 (II) *$(id)B \in \overline{(U' \rtimes S) \cdot w_P B}$.*
 (III) *If $f : \text{Pet}_n \rightarrow \mathbb{R}$ is any $U' \rtimes S$ -invariant, upper-semicontinuous function on Pet_n , then it is maximized at $(id)B$.*

Proof. (I) is Lemma 4.2 parts (I) and (II) combined with Lemma 2.2(I). In general, if $gB \in BwB/B$ and $gB \neq wB$, then $\lim_{t \rightarrow 0} s(t) \cdot gB = uB$ with $u < w$. (II) just applies this. Finally, (III) holds by part (II), together with the hypotheses on f . \square

Example 6.3. In general, the $U' \rtimes S$ action is not transitive on $\mathcal{J}_{w_P, \text{Pet}_n}$. Consider the 4-dimensional $\mathcal{J}_{w_P, \text{Pet}_6}$ where $P = S_3 \times S_3$. One checks the 3-dimensional subgroup $U'' \subset U'$, obtained by setting $q_1 = q_2 = 0$ in the U' of Example 4.1, fixes any gB in $\mathcal{J}_{w_P, \text{Pet}_6}$. Thus, any $U' \rtimes S$ orbit through $w_P B$ has dimension at most 3 and hence cannot cover $\mathcal{J}_{w_P, \text{Pet}_6}$. \square

Lemma 6.2(III) says that $(id)B$ is the “most singular” on Pet_n . This, and the connection of $\mathcal{N}_{id, \text{Pet}_n}$ to $QH^*(GL_n/B)$ (see Section 3) motivates this case of Problem 1.2:

Problem 6.4. Find a combinatorial formula for either $\text{mult}_{id}(\text{Pet}_n)$ and/or $h_{id, \text{Pet}_n}(\eta)$.

The sequence of multiplicities at this point begins:

$$\text{mult}_{123.B}(\text{Pet}_3) = 2, \text{mult}_{1234.B}(\text{Pet}_4) = 4, \text{mult}_{12345.B}(\text{Pet}_5) = 12, \text{mult}_{123456.B}(\text{Pet}_6) = 38, \dots$$

In addition, the sequence of h -polynomials begins:

$$h_{123.B, \text{Pet}_3}(\eta) = 1 + \eta, \quad h_{1234.B, \text{Pet}_4}(\eta) = 1 + 2\eta + \eta^2, \quad h_{12345.B, \text{Pet}_5}(\eta) = 1 + 3\eta + 4\eta^2 + 3\eta^3 + \eta^4, \\ h_{123456.B, \text{Pet}_6}(\eta) = 1 + 4\eta + 8\eta^2 + 10\eta^3 + 8\eta^4 + 4\eta^5 + 2\eta^6 + \eta^7, \dots$$

Question 6.5. Are the coefficients of $h_{g, \text{Pet}_n}(\eta)$ unimodal, or more strongly, log-concave?

While unimodality might be true of h -polynomials of Schubert varieties, log-concavity is known not to be true [LiYon10b, Example 2.2]. These conjectures are not known to follow for any obvious reasons, even if we knew $\text{gr}_{m_{gB}} \mathcal{O}_{gB, \text{Pet}_n}$ to be Cohen-Macaulay.

The following example contrasts with Conjecture 7.4 concerning Richardson varieties:

Example 6.6. $\text{Spec}(\text{gr}_{m_{gB}} \mathcal{O}_{gB, \text{Pet}_n})$ need not be reduced. This occurs for $2134 \cdot B \in \text{Pet}_4$ where the ideal defining the tangent cone is $I'_{2134 \cdot B, \text{Pet}_4} = \langle z_{32} - z_{41}, z_{42}, z_{41}^2 \rangle$.

w_P	$k_{w_P, \text{Pet}_n}(\chi)$	$h_{w_P, \text{Pet}_n}(\eta)$
1234	$(1 - \chi^3)^2(1 - \chi^4)$	$1 + 2\eta + \eta^2$
2134	$(1 - \chi^2)(1 - \chi^3)(1 - \chi^4)$	$1 + \eta$
1324	$(1 - \chi^2)^2(1 - \chi^4)$	$1 + \eta$
1243	$(1 - \chi^2)(1 - \chi^3)(1 - \chi^4)$	$1 + \eta$
2143	$(1 - \chi^3)^2(1 - \chi^2)$	$1 + \eta$
3214	$(1 - \chi^{-1})(1 - \chi^2)(1 - \chi^3)$	1
1432	$(1 - \chi^{-1})(1 - \chi^2)(1 - \chi^3)$	1
4321	$(1 - \chi^{-1})^2(1 - \chi^{-2})$	1

TABLE 1. K -polynomials and h -polynomials at $w_P B \in \text{Pet}_4$

Conjecture 6.7. $\text{gr}_{\mathfrak{m}_{gB}} \mathcal{O}_{gB, \text{Pet}_n}$ is Cohen-Macaulay and thus $h_{g, \text{Pet}_n}(\eta) \in \mathbb{N}[\eta]$. The function $f : \text{Pet}_n \rightarrow \mathbb{R}$ defined by $gB \mapsto$ “coefficient of η^i in $h_{g, \text{Pet}_n}(\eta)$ ”, is upper-semicontinuous on Pet_n .

M. Kummini pointed out that the asymmetry of the coefficients of $h_{123456.B, \text{Pet}_6}(\eta)$ computed above implies that $\text{gr}_{\mathfrak{m}_{123456.B}} \mathcal{O}_{123456.B, \text{Pet}_6}$ is not Gorenstein.

For a point $p \in X$, the Cohen-Macaulayness of the local ring $\mathcal{O}_{p, X}$ is necessary but not sufficient for $\text{gr}_{\mathfrak{m}_p} \mathcal{O}_{p, \text{Pet}_n}$ to be Cohen-Macaulay. That Cohen-Macaulayness implies the positivity of the h -polynomial is [BruHer93, Corollary 4.1.10]. While Hilbert-Samuel multiplicity is upper-semicontinuous, the coefficients of the h -polynomial need not be.

Using (P.V) we verified the Cohen-Macaulayness claim of Conjecture 6.7 for $n \leq 4$ and all but two cases of $n \leq 5$, as well as in many other instances. Table 1 is also consistent with the upper-semicontinuity claim.

7. ANALYSIS OF ADDITIONAL TORUS INVARIANT VARIETIES

Conjecture 6.7 is an analogue of a conjecture stated for Schubert varieties in [LiYon10b, Section 1.2]. In regards to the second meta-question of Section 1.1, we briefly explore the prevalence of the properties of Conjecture 6.7, by examining some other $X \subseteq GL_n/B$.

7.1. Hessenberg varieties and Springer fibers. Fix $H \in M_{n \times n}$ and $\mathfrak{h} : \mathbb{N} \rightarrow \mathbb{N}$. Define $\text{Hess}(H, \mathfrak{h}) = \{F_\bullet \in GL_n/B : H \cdot F_i \subseteq F_{\mathfrak{h}(i)}\}$. These were studied by [DeMProSha92]. The definition we use is stated in [Tym06]. Two families of Hessenberg varieties arise from the cases that H is regular semisimple (eigenvalues of H are distinct and nonzero), and that H is regular nilpotent (this includes Pet_n). In the former case, $\text{Hess}(H, \mathfrak{h})$ is smooth. However, $\text{Hess}(H, \mathfrak{h}) = \emptyset$ unless $\mathfrak{h}(i) \geq i$ and $\mathfrak{h}(i+1) \geq \mathfrak{h}(i)$. When \mathfrak{h} satisfies these conditions, \mathfrak{h} is called a **Hessenberg function**. $\text{Hess}(H, \mathfrak{h})$ is a **Hessenberg variety** if \mathfrak{h} is a Hessenberg function.

Generators for $I_{g, \text{Hess}(H, \mathfrak{h})}$ (up to radical) can be obtained by a modification of the construction of the generators for I_{g, Pet_n} described in Section 1.2 and Example 2.3. (Small examples exist where these “obvious” generators do not define a radical ideal.) The proof of Theorem 1.6 likely generalizes to show that if H is regular nilpotent, and \mathfrak{h} is a Hessenberg function, each patch ideal of $\text{Hess}(H, \mathfrak{h})$ is a local complete intersection.

We checked that the Cohen-Macaulay claim of Conjecture 6.7 holds for all regular nilpotent Hessenberg varieties for $n \leq 4$, most cases of $n \leq 5$ and a few cases of $n \leq 6$. Also, we made some checks of the upper-semicontinuity claim for $n \leq 4$.

Another example of Hessenberg varieties is the **Springer fiber** associated to a partition λ of n . Define $\text{Springer}_\lambda = \{F_\bullet \in GL_n/B : H \cdot F_i \subset F_i \text{ for all } 1 \leq i \leq n\}$, where H is nilpotent matrix of Jordan type λ . A key fact is that the irreducible components Springer_T of Springer_λ are indexed by standard Young tableaux T of shape λ . See, e.g., [FreMel10, PerSmi10] and the references for results on singularities of these components.

We do not know if Conjecture 6.7 holds for Springer_T 's. It is false for Springer_λ in general. For Springer_λ , we checked the Cohen-Macaulay claim of Conjecture 6.7 for $n \leq 5$, and certain “larger” cases such as $\lambda = (3, 2, 1)$ and $\lambda = (2, 2, 1, 1)$ (the latter case was the first known to have singular irreducible components [Var79]).

Example 7.1. The aforementioned singular component of $\text{Springer}_{(2,2,1,1)}$ found by [Var79] is the unique singular one in that Springer fiber [Fre10]. It has eight singular points from $\text{Springer}_{(2,2,1,1)} \cap (GL_6/B)^T$, corresponding to the permutations: 135624, 135642, 136524, 136542, 315624, 315642, 316524, 316542. On this component, these all have Hilbert-Samuel multiplicity 6, the same h -polynomial $1 + 4\eta + \eta^2$, and Gorenstein tangent cones. \square

7.2. Richardson varieties. The **Richardson variety** is defined by $X_u^v = \overline{BuB/B} \cap \overline{B_{-v}B/B}$. This is empty unless $v \leq u$ and is of dimension $\ell(u) - \ell(v)$ otherwise, where $\ell(u)$ is the Coxeter length of u . It is the **Schubert variety** $X_u = \overline{BuB/B}$ when $v = id$ and the **opposite Schubert variety** $X^v = \overline{B_{-v}B/B}$ when $u = w_0$. Richardson varieties are of importance in connection to Schubert calculus since $[X_u^v] = [X_u] \cup [X_{w_0v}] = \sum_{x \in S_n} C_{u, w_0v}^x [X_x] \in H^*(GL_n/B; \mathbb{Z})$, where C_{u, w_0v}^x is a **generalized Littlewood-Richardson coefficient**.

From the definitions, $wB \in X_u^v$ if and only if $v \leq w \leq u$ (in Bruhat order). In this case, it is known X_u^v is reduced, irreducible and Cohen-Macaulay; see, e.g., [Bri05].

The decomposition (P.III) is, in this setting:

$$\begin{aligned} X_u^v &= \coprod_w (BwB/B \cap X_u^v) = \coprod_w BwB/B \cap (\overline{BuB/B} \cap \overline{B_{-v}B/B}) \\ &= \coprod_w (BwB/B \cap \overline{BuB/B}) \cap X^v = \coprod_{w \leq u} (BwB/B) \cap X^v. \end{aligned}$$

Each $\mathcal{J}_{w, X_u^v} = (BwB/B) \cap X^v$ is a **Kazhdan-Lusztig variety**. This variety is isomorphic, up to crossing by an affine space, to the patch of the opposite Schubert variety X^v [KazLus79, Lemma A.4]. It is singular in general. The maximal torus T in $B \cap B_{-}$ acts on each \mathcal{J}_{w, X_u^v} , contracting each point towards wB . Hence wB is the most singular point on that strata, in the sense of Lemma 2.2(II).

We want generators for I_{w, X_u^v} . Let $J_{w, u}$ be the ideal in $\mathbb{C}[Z^{(w)}]$ generated by all size $1 + r_{ij}^u$ minors of the southwest $(n - i + 1) \times j$ submatrix of $Z^{(w)}$. Here r_{ij}^u is the rank matrix that counts the number of 1's southwest of matrix coordinate (i, j) in the permutation matrix for u . Let $J^{w, v}$ be defined by $1 + \bar{r}_{ij}^v$ minors of the northwest $i \times j$ submatrix where \bar{r}_{ij}^v is the number of 1's northwest of matrix coordinate (i, j) in the permutation matrix for v .

The following is probably folklore, and is implicit in [KazLus79, Lemma A.4]:

Proposition 7.2. $I_{w, X_u^v} = J_{w, u} + J^{w, v}$.

Proof. Fulton [Ful91] showed that if $\pi : GL_n \rightarrow GL_n/B$ is the natural map, then the ideal defining $\pi^{-1}(X_u)$ is scheme-theoretically generated by the size $1 + r_{ij}^u$ minors of the southwest $(n - i + 1) \times j$ submatrix. Thus it follows from (2.2) that $I_{w, X_u} = J_{w, u}$. Similarly,

$I_{w,X^v} = J^{w,v}$. Since I_{w,X_u} scheme-theoretically cuts out an open neighbourhood of wB in X_u and I_{w,X^v} does the same inside X^v , then $I_{w,X_u} + I_{w,X^v}$ does it for $wB \in X_u \cap X^v$. Now $I_{w,X_u} + I_{w,X^v}$ is radical since $X_u \cap X^v$ is reduced. The result then follows. \square

Problem 7.3. Find an explicit Gröbner basis for I_{w,X_u^v} .

Computation suggests that there is a Gröbner basis with a squarefree monomial ideal.

Conjecture 7.4. The tangent cones of X_u^v are reduced.

Using Proposition 7.2 and (P.V), we verified Conjecture 7.4 and positivity of the h -polynomial exhaustively at T -fixed points, for $n \leq 5$, and for many random singular cases for $6 \leq n \leq 9$. We also checked the tangent cones at T -fixed points are Cohen-Macaulay, for $n \leq 4$. For Grassmannian Richardson varieties, these claims follow from [KreLak04, Remark 7.6.6]. These facts, combined with the results/checks made for the Schubert case [LiYon10b], leave us reasonably convinced. However, we have not made substantive checks of upper-semicontinuity of the coefficients of $h_{g,X_u^v}(\eta)$ in this setting.

7.3. Peterson-Schubert varieties. Let $R_w = \text{Pet}_n \cap X_w$. This scheme (it is reducible in general) is S -invariant since S acts on both Pet_n and X_w .

Lemma 7.5. R_{w_P} is irreducible and of dimension $n - k$, where k is the number of blocks of w_P (i.e. the k of Equation (1.1)).

Proof. By Lemma 4.2(VI), the permutation matrix of w_P has block matrix form

$$(7.1) \quad w_P = C^{(1)} \oplus C^{(2)} \oplus \dots \oplus C^{(k)}$$

where $C^{(j)}$ is an $i_j \times i_j$ block matrix that is the “reverse identity matrix” of that size. We claim (and this is a well-known fact) that

$$(7.2) \quad GL_{i_1}/B_{i_1} \times GL_{i_2}/B_{i_2} \times \dots \times GL_{i_k}/B_{i_k} \cong X_{w_P} \cong P/B,$$

where $i_1 + \dots + i_k = n$, cf. (1.1) and B_{i_j} is the Borel subgroup of upper triangular matrices inside $GL(i_j)$. To see this, let $F_{\bullet}^{(t)} \in GL_{i_t}/B$, $1 \leq t \leq k$ and consider the map $\phi : GL_{i_1}/B_{i_1} \times GL_{i_2}/B_{i_2} \times \dots \times GL_{i_k}/B_{i_k} \rightarrow X_{w_P}$ defined by

$$\phi : (F_{\bullet}^{(t)})_{t=1}^k \mapsto (F_{\bullet}^{(1)}, \mathbb{C}^{i_1} \oplus F_{\bullet}^{(2)}, \mathbb{C}^{i_1} \oplus \mathbb{C}^{i_2} \oplus F_{\bullet}^{(3)}, \dots, \mathbb{C}^{i_1} \oplus \mathbb{C}^{i_2} \oplus \dots \oplus \mathbb{C}^{i_{k-1}} \oplus F_{\bullet}^{(k)}) \in GL_n/B,$$

where $V \oplus F_{\bullet} := V \oplus F_1 \subset V \oplus F_2 \subset \dots$. We also made the natural identification of $F_j^{(1)} \subseteq \mathbb{C}^{i_1}$ with a subspace of $\mathbb{C}^n \cong \bigoplus_{j=1}^k \mathbb{C}^{i_j}$, etc.

Clearly, the image of ϕ is X_{w_P} and that ϕ is an isomorphism. Thus (7.2) holds.

The map ϕ restricts to $\psi : \text{Pet}_{i_1} \times \dots \times \text{Pet}_{i_k} \rightarrow GL_n/B$. The image sits inside X_{w_P} since that is true of ϕ . For each t we have $N \cdot (\mathbb{C}^{i_1} \oplus \mathbb{C}^{i_2} \oplus \dots \oplus \mathbb{C}^{i_t}) \subset \mathbb{C}^{i_1} \oplus \mathbb{C}^{i_2} \oplus \dots \oplus \mathbb{C}^{i_t}$ (where $\mathbb{C}^{i_1} \oplus \mathbb{C}^{i_2} \oplus \dots \oplus \mathbb{C}^{i_t}$ is interpreted as a subspace of \mathbb{C}^n in the manner indicated above). Let $N^{(j)}$ be the standard regular nilpotent matrix for $GL(i_j)$. Then by assumption, $N^{(j)} \cdot F_r^{(j)} \subset F_{r+1}^{(j)}$. Although of course the block matrix $N^{(1)} \oplus N^{(2)} \oplus \dots \oplus N^{(k)}$ is not equal to N , it is easy to see from the definitions that the image of ψ is inside Pet_n . Hence,

$$(7.3) \quad \psi(\text{Pet}_{i_1} \times \text{Pet}_{i_2} \times \dots \times \text{Pet}_{i_k}) \subseteq R_{w_P}$$

(as sets). It remains to demonstrate the reverse containment. Now,

$$R_{w_P} = X_{w_P} \cap \text{Pet}_n = \coprod_{w \leq w_P} (BwB/B) \cap \text{Pet}_n = \coprod_{w_Q \leq w_P} (Bw_QB/B) \cap \text{Pet}_n = \coprod_{w_Q \leq w_P} U_Q w_Q B/B,$$

where we have used Lemma 4.2 (VI) and (IV). We have also used the fact that $w_Q B \in R_{w_P}$ if and only if $w_Q \leq w_P$. In this case, the permutation matrix for w_Q takes the form

$$(7.4) \quad w_Q = \tilde{C}^{(1)} \oplus \dots \oplus \tilde{C}^{(k)}$$

where $\tilde{C}^{(j)}$ occupies the same position as the block $C^{(j)}$. Thus, the desired containment holds if we can show $U_Q w_Q B \subseteq \psi(\text{Pet}_{i_1} \times \text{Pet}_{i_2} \times \dots \times \text{Pet}_{i_k})$ for each $w_Q \leq w_P$. However, this is clear from (7.4) and the definition of ψ .

Hence R_{w_P} is irreducible, as each factor on the lefthand side of (7.3) is irreducible (and we are working over \mathbb{C}). The dimension claim for R_{w_P} holds since $\dim(\text{Pet}_i) = i - 1$. \square

In view of Lemma 7.5 and Theorem 7.6(I) below, we call R_{w_P} a **Peterson-Schubert variety**; cf. [HarTym11]. We have made checks of the Cohen-Macaulayness property of Conjecture 6.7 for the varieties R_{w_P} when $n \leq 5$.

Referring to (7.4), let $w_Q^{(j)}$ denote the permutation in S_{i_j} that corresponds to the matrix $\tilde{C}^{(j)}$. The following extends Theorem 1.4 and Corollary 1.8.

- Theorem 7.6.** (I) R_{w_P} is a reduced, local complete intersection (and hence Cohen-Macaulay and Gorenstein).
 (II) R_{w_P} is singular if and only if w_P contains the pattern 321.
 (III) The singular locus of R_{w_P} is given by $\text{Sing}(R_{w_P}) = \coprod_{w_Q} \mathcal{J}_{w_Q, R_{w_P}}$, where $w_Q \leq w_P$ and at least one of the $w_Q^{(j)}$ contains either of the patterns 123 or 2143.

Proof. (I): We want to show that the neighbourhood of $gB \in R_{w_P}$ given by $\mathcal{N}_{g, X_{w_P}} \cap \mathcal{N}_{g, \text{Pet}_n}$ is reduced and a complete intersection. Since $gB \in \text{Pet}_n$, then $gB = bw_Q B$ for some $b \in U_Q$, by Lemma 4.2(IV). By the discussion of Section 2, we have that $\mathcal{N}_{g, \text{Pet}_n} = \mathcal{N}_{w_Q, \text{Pet}_n}$ and similarly $\mathcal{N}_{g, X_{w_P}} = \mathcal{N}_{w_Q, X_{w_P}}$. Therefore, $\mathcal{N}_{g, X_{w_P}} \cap \mathcal{N}_{g, \text{Pet}_n}$ and $\mathcal{N}_{w_Q, X_{w_P}} \cap \mathcal{N}_{w_Q, \text{Pet}_n}$ are equal. So, it suffices to assume $g = w_Q$.

By the definition of $R_{w_P} = X_{w_P} \cap \text{Pet}_n$ we have $I_{w_Q, R_{w_P}} = I_{w_Q, X_{w_P}} + I_{w_Q, \text{Pet}_n}$. Referring to (7.4), we have the finer block decomposition

$$(7.5) \quad \tilde{C}^{(j)} = D^{(j,1)} \oplus \dots \oplus D^{(j,f)}$$

where each $D^{j,g}$ is a reverse identity matrix. That is, $\tilde{C}^{(j)} = C^{(j)}$ if and only if $f = 1$.

Claim 7.7. The following equality of ideals holds

$$(7.6) \quad I_{w_Q, X_{w_P}} + I_{w_Q, \text{Pet}_n} = I_{w_Q, X_{w_P}} + \sum_{j=1}^k \hat{I}_{w_Q^{(j)}, \text{Pet}_{i_j}},$$

where $\hat{I}_{w_Q^{(j)}, \text{Pet}_{i_j}}$ is the patch ideal for $w_Q^{(j)} \in \text{Pet}_{i_j}$, and where we have embedded the ring $\mathbb{C}[Z^{(w_Q)}]$ into $\mathbb{C}[Z^{w_P}]$ using the variables of $Z^{(w_P)}$ in the region occupied by $\tilde{C}^{(j)}$ in the obvious manner.

Proof. Recall that by Proposition 7.2, the generators of $I_{w_Q, X_{w_P}}$ consist of all variables in the region below the blocks $C^{(j)}$. Next, consider any generator $g_{k', j'}$ of I_{w_Q, Pet_n} as described by (1.9). Suppose j' is one of the columns $a, a+1, \dots, b$ occupied by $C^{(j)}$. Note that by the construction of $g_{k', j'}$, row k' cannot be strictly above the topmost row occupied by $C^{(j)}$ (since there is a 1 to the left of column j' and in row k'). There are two cases:

Case 1: $j' \leq b - 2$: (i) First suppose row k' is strictly south of the bottom row occupied by $C^{(j)}$. Then since $j' \leq b - 2$, we see $g_{k',j'} = \alpha_{j',1}z_{k',1} + \cdots + \alpha_{j',j'+1}z_{k',j'+1} - z_{k'+1,j'} \in I_{w_Q, X_{w_P}}$ since $z_{k',1}, \dots, z_{k',j'+1}, z_{k'+1,j'} \in I_{w_Q, X_{w_P}}$.

(ii) If row k' is one of the rows occupied by $C^{(j)}$, then each variable of $Z^{(w_Q)}$, in that row and in columns $1, 2, \dots, a - 1$, is in $I_{w_Q, X_{w_P}}$. Therefore, one can decompose $g_{k',j'} = \bar{g}_{k',j'} + g_{k',j'}^\vee$ where $\bar{g}_{k',j'} \in I_{w_Q, X_{w_P}}$ and $g_{k',j'}^\vee$ is one of the generators (1.9) of $\hat{I}_{w_Q^{(j)}, \text{Pet}_{i_j}}$. Conversely, every generator of $\hat{I}_{w_Q^{(j)}, \text{Pet}_{i_j}}$ gives rise to a generator of I_{w_Q, Pet_n} in this manner.

Case 2: $j' = b - 1$ or $j' = b$: By the construction of $g_{k',j'}$, we see that row k' must be strictly south of the rows occupied by $C^{(j)}$. Then $g_{k',j'} \in I_{w_Q, X_{w_P}}$, just as in Case 1(i).

The equality (7.6) then follows from these considerations. \square

X_{w_P} is smooth and hence a local complete intersection. By Corollary 1.8 each Pet_{i_j} is a complete intersection. Since Claim 7.7 expresses $I_{w_Q, X_{w_P}} + I_{w_Q, \text{Pet}_n}$ as a sum of ideals on disjoint variable sets, each ideal being a radical, complete intersection, (I) holds.

(II): First suppose that w_P contains the pattern 321. Hence, one of the factors Pet_{i_j} in

$$X_{w_P} \cap \text{Pet}_n \cong \text{Pet}_{i_1} \times \text{Pet}_{i_2} \times \cdots \times \text{Pet}_{i_k}$$

is isomorphic to the singular Pet_m for some $m \geq 3$. Conversely, suppose that R_{w_P} is singular. Therefore one of the factors Pet_{i_j} is singular (since a product of smooth varieties is smooth). Since Pet_2 is smooth, we deduce that $i_i \geq 3$. Hence w_P contains 321.

(III): The discussion of (I), and specifically (7.6), imply

$$\mathcal{N}_{g, R_{w_P}} \cong \mathcal{N}_{g^{(1)}, \text{Pet}_{i_1}} \times \mathcal{N}_{g^{(2)}, \text{Pet}_{i_2}} \times \cdots \times \mathcal{N}_{g^{(k)}, \text{Pet}_{i_k}},$$

where $g = bw_Q$ for $b \in U_Q = B^{(1)} \oplus B^{(2)} \oplus \cdots \oplus B^{(k)}$ where each $B^{(j)} \in U_{Q^{(j)}}$ and $Q^{(j)}$ is the parabolic subgroup of GL_{i_j} corresponding to $w_Q^{(j)}$. Writing $b = b_1 \oplus b_2 \oplus \cdots \oplus b_k$ where $b_j \in B^{(j)}$, in the above isomorphism we set $g^{(j)} = b_j \cdot w_Q^{(j)}$. Now $\mathcal{N}_{g, R_{w_P}}$ is smooth if and only if each $\mathcal{N}_{g^{(j)}, \text{Pet}_{i_j}}$ is smooth. Then the claim follows from Theorem 1.4. \square

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